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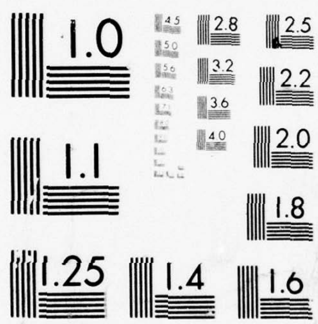

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MODELS OF HIERARCHAL REPLACEMENT

TECHNICAL REPORT

N.L. JOHNSON, SAMUEL KOTZ

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DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL
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MODELS OF HIERARCHAL REPLACEMENT

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1. Introduction

This note is partially motivated by a paper by Krakowski (1973) in which he describes the following process. At time zero an individual is chosen at random from a population A of new (age zero) individuals. When this individual fails, it is replaced by an individual chosen at random from the survivors among a population B created (i.e. at age zero) at the same time as A. Krakowski found an expression for the distribution of failure time of this replacement.

We have developed this model to allow for more than one stage of replacement, with different populations providing the replacement at different stages. The condition that replacements are chosen from populations aging simultaneously is retained.

Furthermore, (motivated by B.C. Arnold's (1975) work on the classification of multivariate exponential distributions based on hierarchal successive damages which extends the well known fatal shock model for Marshall and Olkin's (1967) bivariate exponential distribution) we have imposed a hierarchal structure on the replacement scheme in two different ways, corresponding to two different replacement strategies for a number of elements which are in service simultaneously.

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In all these cases we are primarily interested in the final failure time, but we also derive the joint distribution of failure (and replacement) times at intermediate stages.

2. Hierarchal Replacement Systems (Procedures)

We will now describe the two replacement systems (procedures) which we will study, first descriptively and then more formally introducing a special notation.

System (Procedure) 1: We suppose that there are k "positions" each occupied by an "element" from a given set A. Initially all k positions are serviced by a single "component." The components are individuals which are assumed to be chosen at random from a population with survival distribution function (SDF) $S_0(t)$. When a certain component fails, it is replaced by s_1 components each servicing a disjoint subset of $k_1,\ldots,k_{s_1}$ of the k positions ($k_1+k_2+\ldots+k_{s_1} = k$). These new components are chosen at random from a population which was originated ("born") at the same time as the original component, but has survival distribution $S_1(t)$.

Each of the s_1 subsets now has the same status as the original set of k elements. The i-th subset is divided into s_{i2} sub^2-sets, and when the component servicing this subset fails it is replaced by s_{i2} components chosen at random from a population with initial survival distribution function (SDF) $S_2(t)$ which was (similarly to components chosen previously from the SDF $S_1(t)$) new at time zero.

The process continues until each position is assigned a different servicing component - corresponding to sub^r-sets - and containing a *single* individual.

We are interested in (i) the times at which original positions receive a component not shared with any other position and (ii) the failure times of these components.

The model we have described above might be used to represent the failure times of elements of an organism in which replacement of damaged elements becomes progressively more specialized. The first failure of an individual in a sub^r(single individual)-set might be regarded as corresponding to the failure of the whole system.

System (Procedure) 2: This system is one possible inverse of System (Procedure) 1. We suppose that initially (at time zero) we have k elements (chosen at random from a population with SDF $S_r(t)$) grouped in sub^{r-1}sets (as in the penultimate stage of System (Procedure) 1). When *any* element in a sub^{r-1}set fails, *all* elements in that sub^{r-1}set are replaced, in effect, by *one* servicing component chosen at random from a population with SDF $S_{r-1}(t)$. We will call such a component a *second stage component*. The sub^{r-1}sets are grouped in sub^{r-2}sets. A sub^{r-2}set may contain some sub^{r-1}sets serviced by a common (second stage) component, and some in which the original elements still survive (not having failed as yet). Should any of the *second stage* components fail, *all* elements in the sub^{r-2}set to which it belongs are replaced, in effect, by a third stage component servicing *every* position in the sub^{r-2}set. The sub^{r-2}sets are grouped in sub^{r-3}sets, and so on. Generally, a sub^{r-s}set can contain sub^{r-s+j}sets each having a common servicing component, for j=1,2,...,s (a sub^{r}set being - as indicated above - a single element). When any component servicing a sub^{r-s+1}set fails, all positions in the sub^{r-s}set to which it belongs are serviced by a common ((s+1)-stage) component, chosen from a

population with SDF $S_{r-s}(t)$.

As in System (Procedure) 1, all components are new at time zero.

Eventually we should reach a situation where we have some (r-1)-th stage elements present. These are not members of any subset, and when one of them fails, *all* elements (of all stages) have to be replaced.

This model corresponds to progressively *less* specialized replacement. The time of failure of the first (r-1)-th stage element to fail might be regarded at the time of failure of the *organism as a whole*.

3. Notation

We now develop a notation which will enable us to express the operation of Procedures (Systems) 1 and 2 more precisely. The formalization devised below can be applied with equal facility to some other (obvious) variants of the two extreme models described in the previous Section. An example will be given at the end of this Section.

It is first necessary to establish identification of various sets of elements. We suppose that the k positions are initially split up into s_1 "first-level" (sub)sets; that the i-th first-level set is split up into $s_2(i)$ second-level (sub^2)sets; that the j-th second level set in the i-th first level set is split up into $s_3(i,j)$ third level (sub^3)sets, and so on. Generally, we denote the i_h-th h-th level set in the i_{h-1}-th (h-1) level set in . . . in the i_1-th first level set by $S_h\ (i_1,i_2,\ldots,i_h)$ for $i_1 = 1,\ldots,s_1$; $i_2 = 1,\ldots,s_2(i_1)$; $i_3 = 1,\ldots,s_3(i_1,i_2)\ldots$, and $i_h = 1,\ldots,$ $s_h\ (i_1,i_2,\ldots,i_{h-1})$, where h runs from 1 to r.

Figure 1 sets out the corresponding subdivision diagrammatically. Note that in System (Procedure) 1 the "level" is the same as the "stage" but in System (Procedure) 2 the relation between stages and levels is given by:

$$\text{stage} = r+1 - (\text{level}). \tag{1}$$

Finally, we denote the number of elements in $S_h\ (i_1,i_2,\ldots,i_h)$ by $k_h(i_1,i_2,\ldots,i_h)$. Clearly the sum of $k_h(\circ)$ over all combinations of values of the arguments is equal to k, (the number of the positions) that is

$$\sum_{i_1} \circ\circ\circ \sum_{i_h} k_h(i_1,i_2,\ldots,i_h) = k\ . \tag{2}$$

Also it follows from the definition that

$$\sum_{i_h} k_h(i_1,i_2,\ldots,i_h) = k_{h-1}(i_1,i_2,\ldots,i_{h-1}). \tag{3}$$

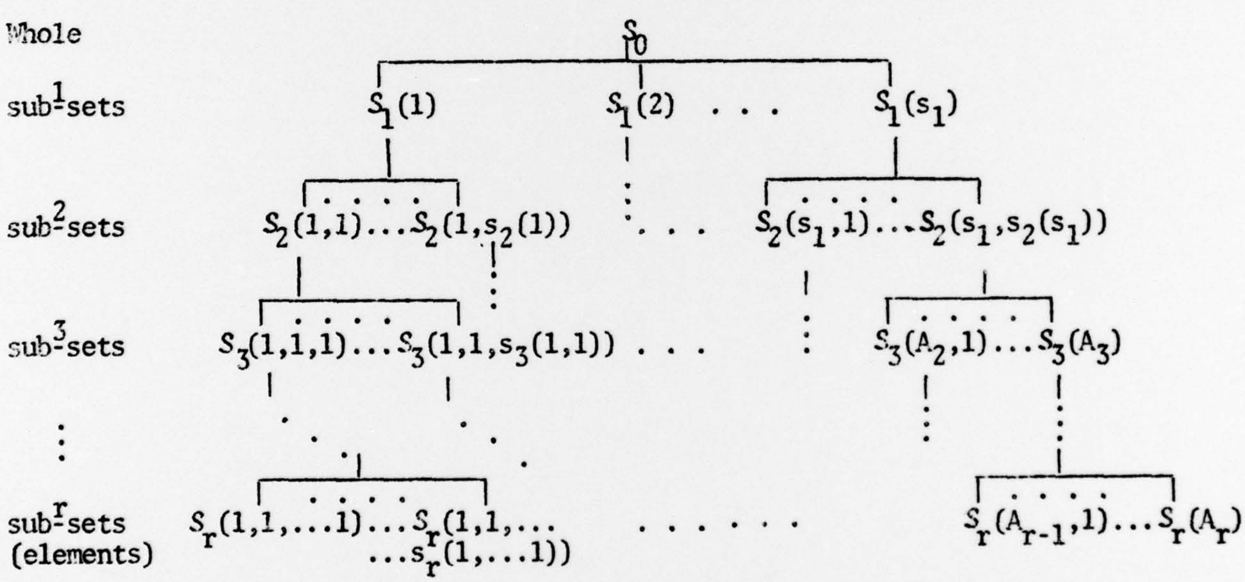

FIGURE 1. Schematic representation of levels and stages in hierarchal replacement models.

Note: A_j is defined by the recurrence relation

$$A_j = A_{j-1}, s_j(A_{j-1})$$

with $A_1 = s_1$.

Using this notation, System (Procedure) 1 can be defined succinctly as follows:

(i) Initially (at time zero) we have a set S_0 of k elements, one in each of k positions, all serviced by a single component randomly chosen from a population with survival distribution function (SDF) $S_0(t)$;

(ii) When the servicing component for a sub^j-set $S_j(\circ)$ ($j=1,\ldots,r$) fails, it is replaced by $s_{j+1}(\circ)$ different servicing components, one for each of the sub^{j+1}-sets in $S_j(\circ)$. These servicing components are chosen at random from the *survivors* in a population with SDF $S_{j+1}(t)$ which was new (i.e. originated) at time zero.

System (Procedure) 2 can now be defined as follows:

(i) Initially (at time zero) we have k individual elements ("sub^r-sets") each chosen at random from a population with SDF $S_r(t)$.

(ii) When the servicing component for a sub^j-set $S_j(\circ)$ ($j=r, r-1,\ldots,1$) fails, the sub^{j-1}-set $S_{j-1}(\circ)$ to which it belongs is assigned to a common servicing component chosen at random from the survivors of a population with SDF $S_{j-1}(t)$ which was new at time zero. This component now replaces all components previously servicing elements belonging to $S_{j-1}(\circ)$.

We may have a model in which the two systems are mixed. A simple case would be when, initially, replacement is according to System 1, but after reaching the stage of sub^h-sets with separate servicing components (h being specified) a failure of *any* of these components results in assignment of a newly selected servicing component to the whole of the sub^{h-1}-set to which the corresponding sub^h-set belongs (as in System 2). Subsequently, System 1 is followed throughout (so that the next time a sub^h-set servicing

component fails each of the sub^{h+1}-sets contained in the sub^{h}-set is assigned a different freshly chosen servicing component).

We could denote the sets involved in the first period (when System 1 is operating) by $S_j(\cdot)$, and those thereafter, by $S_j'(\cdot)$, starting, of course, with $S_{h-1}'(\cdot)$.

Further generalizations to a number of changes from System 1 to System 2 and conversely, could be accommodated by including greater detail in the superfix of $S(\cdot)$, or by using prefixes. For example if there were changes from System 1 to 2 on failure of $h_1, h_2, \ldots, h_q$ level components, and converse changes at $g_1, g_2, \ldots, g_q$ with

$$g_1 < h_1 < g_2 < h_2 < \ldots < g_q < h_q$$

(no change in system at second failure at an h_ℓ stage) the set S_j, for $g_i < j < h_i$ reached after the change (from System 2 to 1) at the g_i-th level would be denoted by ${}_{g_i}S_j(\cdot)$.

4. Distributions of Failure Times: System (Procedure) I.

We first consider the failure time distribution for the servicing component of the sub^{j}-set S_j (1,1,...,1). This failure time will be denoted by T_j. (If necessary, the more explicit notation T_j (1,1,...,1) could be used.) The sub^{j}set S_j (1,1,...,1) contains

$$S_{1\ldots1}^{(j)} = \sum_{a_{j+1}} \cdots \sum_{a_r} S_r(1,\ldots,1,a_{j+1},\ldots a_r) \tag{4}$$

components.

Let $\{Y_j\}$ be a sequence of independent random variables $(j=0,1,\ldots)$ with SDF's $S_j(t)$.

When S_0 fails at time y_0 (according to SDF $S_0(y_0)$), the replacement servicing component (for the subset $S_1(1)$ containing S_j (1,1,...,1)) has a

conditional failure time distribution as that of random variable Y, truncated from below at y_0. The conditional SDF is

$$\{S_1(y_0) - S_1(t)\}/S_1(y_0) \qquad (t \geq y_0) \tag{5}$$

and the corresponding conditional density is

$$f_1(t)/S_1(y_0) \qquad (t \geq y_0) \tag{6}$$

where $f_j(t) = -dS_j(t)/dt$.

Hence (with $y_0, y_1, \ldots, y_{j-1}, y_j$ denoting failure times, at successive stages, of the servicing components, for the whole set, subset,...,sub^{j-1}set and sub^{j}set (which is just $S_j(1,1,\ldots,1)$) respectively containing the subset $S_j\ (1,1,\ldots,1)$) we have

$$\Pr[T_j \leq \tau] = \int_0^{\tau} \int_0^{y_j} \cdots \int_0^{y_2} \int_0^{y_1} f_0(y_0) \frac{f_1(y_1)}{S_1(y_0)} \cdots \frac{f_j(y_j)}{S_j(y_{j-1})} dy_0 dy_1 \ldots dy_j$$

$$= \int_0^{\tau} \int_0^{y_j} \cdots \int_0^{y_2} \int_0^{y_1} \frac{\Pi_{i=0}^{j} f_i(y_i)}{\Pi_{i=1}^{j} S_j(y_{i-1})} dy_0 dy_1 \ldots dy_j \tag{7}$$

$$= \int_0^{\tau} \int_0^{y_j} \cdots \int_0^{y_2} \int_0^{y_1} \left\{ \prod_{i=0}^{j-1} \frac{f_i(y_i)}{S_{i+1}(y_i)} \right\} f_j(y_j) dy_0 dy_1 \ldots dy_j \ . \tag{7'}$$

For j = 1 we have

$$\Pr[T_1 \leq \tau] = \int_0^{\tau} f_1(y_1) \int_0^{y_1} \{f_0(y_0)/S_1(y_0)\} dy_0 dy_1$$

$$= \int_0^{\tau} \{f_0(y_0)/S_1(y_0)\} \int_{y_0}^{\tau} f_1(y_1) dy_1 dy_0$$

$$= \int_0^{\tau} \{f_0(y_0)/S_1(y_0)\}\{S_1(y_0) - S_1(\tau)\} dy_0$$

$$= 1 - S_0(\tau) - S_1(\tau) \int_0^{\tau} \{f_0(y_0)/S_1(y_0)\} dy_0$$

or

$$\Pr[T_1>\tau] = S_0(\tau) - S_1(\tau)\int_0^\tau \frac{dS_0(y_0)}{S_1(y_0)} \tag{8}$$

in agreement with Krakowski's formula (1.1).

It is interesting to note that the distribution given by (8) differs from the distribution of age at failure of a randomly chosen individual from $S_1(\circ)$, given that it exceeds age at failure of a randomly chosen individual from $S_0(\circ)$. The cdf of the latter is

$$\frac{\int_0^\tau f_1(t)F_0(t)dt}{\int_0^\infty f_1(t)F_0(t)dt}, \tag{8a}$$

where $F_i(t) = 1 - S_i(t)$, $i=0,1$.

In (8) we suppose we obtain an observation corresponding to *every* age at death of the individual from $S_0(\circ)$; in (8a) an observation is obtained *only when* the individual from $S_1(\circ)$ survives that from $S_0(\circ)$, so that the larger values of the argument get relatively low weighting. Note also that failure times for any individual servicing component (including that of the *elements* in the final stage) are not independent; therefore the distribution of the first or the last failure time among these components or elements is not easily obtained.

Examples. a) If we suppose that each servicing component (at whatever level) has a Weibull distribution, with SDF

$$S_j(t) = \exp\{-(t/\theta)^c\} \qquad (t\geq 0;\ \theta>0) \tag{9}$$

so that

$$\frac{f_j(t)}{S_{j+1}(t)} = c\theta^{-1}(t/\theta)^{c-1} \tag{10}$$

then (7) becomes

$$\Pr[T_j \le \tau] = (c\theta^{-1})^j \int_0^{\tau}\int_0^{y_j} \cdots \int_0^{y_2}\int_0^{y_1} \{\prod_{i=0}^{j} (y_i/\theta)\}^{c-1} \exp\{-(y_j/\theta)^c\} dy_0 dy_1 \ldots dy_j$$

$$= \theta c^{-j} \int_0^{\tau/\theta}\int_0^{t_j} \cdots \int_0^{t_2}\int_0^{t_1} (\prod_{i=0}^{j-1} t_i)^{c-1} t_j^{c-1} e^{-t_j^c} dt_0 \ldots dt_j$$

$$= \frac{1}{j!} \int_0^{\tau/\theta} t_j^{(j+1)c-1} e^{-t_j^c} dt_j \ . \tag{11}$$

In the case given by formula (11) T_j/θ has a generalized gamma distribution (see, e.g., Johnson and Kotz (1970)). It is distributed as the c-th root of a gamma variable with parameter $(j+1)$.

b) Taking $c = 1$ gives the expression for the special case of common exponential survival distribution functions.

c) If we suppose that

$$S_i(t) = e^{-t/\theta_i} \qquad (t \ge 0;\ \theta_i > 0) \tag{12}$$

with not all θ's equal, then we find

$$\Pr[T_j \le \tau] = (\prod_{i=0}^{j} \theta_i)^{-1} \int_0^{\tau}\int_0^{y_j} \cdots \int_0^{y_2}\int_0^{y_1} \exp\{-\sum_{i=0}^{j} (\theta_i^{-1} - \theta_{i+1}^{-1}) y_i - \theta_j^{-1} y_j\} dy_0 dy_1 \ldots dy_j . \tag{13}$$

Integrals of the form

$$I(a_0, a_1, \ldots, a_j) = \int_0^{\tau}\int_0^{y_j} \cdots \int_0^{y_2}\int_0^{y_1} \exp(-\sum_{i=0}^{j} a_i y_i) dy_0 dy_1 \ldots dy_j \tag{14}$$

satisfy the recurrence relation:

$$I(a_0, a_1, \ldots, a_j) = a_0^{-1}\{I(a_1, a_2, \ldots, a_j) - I(a_0 + a_1, a_2, \ldots, a_j)\} \ . \tag{15}$$

Using

$$I(a_0) = \int_0^{\tau} \exp(-a_0 y_0) dy_0 = a_0^{-1}(1 - e^{-a_0 \tau}) \tag{16}$$

and repeatedly applying (15) we can successively obtain the explicit values of $I(a_0,a_1,\ldots,a_j)$.

Introducing the displacement operator E, such that $E^h f(x) = f(x+h)$ we have

$$I(a_0,a_1,\ldots,a_j) = \Big(\prod_{i=0}^{j} a_i\Big)^{-1} \prod_{i=0}^{j} (1-E^{a_i})\exp(-0.\tau)$$

$$= \Big(\prod_{i=0}^{j} a_i\Big)^{-1}\{1-\sum_{i=0}^{j} E^{a_i}+\sum_{i<i'}\sum E^{a_i+a_{i'}}-\ldots$$

$$+(-1)^{j+1}E^{\Sigma_0^j a_i}]\exp(-0.\tau)$$

$$= \Big(\prod_{i=0}^{j} a_i\Big)^{-1}\{1-\sum_{i=0}^{j} e^{-a_i\tau}+\sum_{i<i'}\sum e^{-(a_i+a_{i'})\tau}-\ldots$$

$$+(-1)^{j+1}e^{-(\Sigma_0^j a_i)\tau}\} . \qquad (17)$$

In System (Procedure) 2, the situation is somewhat more complex. However, the distributions of final failure time can be explicitly derived without much difficulty. For computational simplicity, we will restrict our calculations to the symmetric, balanced case, in which the number of sub^{j}sets in each sub^{j-1}set (i.e. serviced by the same (j-1)-th level component) is the same - s_j - say. In this case $k = \Pi_{j=1}^r s_j$ and the number of sub^{h}sets is $\Pi_{j=1}^h s_j$.

We denote the survival time distribution function for an *h-th level* service component (failing in service) by $S_{(h)}(t)$. This is to be distinguished from the SDF $S_h(t)$ in the population from which the h-th level service components are chosen when a relevant (h+1)-th level component fails. From the symmetry of the hierarchal structure, the SDF for each of the s_h h-th level components to be serviced by any one (h-1) level component is the same, and so the SDF of the first failure time among these s_h h-th level components is

$$\{S_{(h)}(t)\}^{s_h} . \tag{19}$$

Conditional on this time being equal to y_h', say, the SDF of failure time of the (h-1)-th level component, activated at time y_h' is the truncated distribution

$$S_{(h-1)}(t|y_h') = \frac{S_{h-1}(t)}{S_{h-1}(y_h')} \qquad (t \geq y_h') . \tag{20}$$

Averaging over the distribution (19) of first failure time of the relevant h-th level components, we obtain

$$S_{(h-1)}(t) = s_h S_{h-1}(t) \int_0^t \frac{\{S_{(h)}(y)\}^{s_h - 1} f_{(h)}(y)}{S_{h-1}(y)} dy \tag{21}$$

where $f_{(h)}(y) = -dS_{(h)}(y)/dy$.

We also have for the ultimate level r:

$$S_{(r)}(t) = S_r(t) . \tag{22}$$

Applying (22) repeatedly with h = r, r-1,...,1 we eventually obtain an explicit formula for $S_{(0)}(t)$, the SDF of "final failure time."

Although this calculation can be effected numerically in a straightforward way, it often leads to rather complicated algebraic expressions. In the following we derive an expression for $S_{(r-2)}(t)$ where each SDF is a Weibull with the same c, but different θ parameters.

Then, from (21) and (22)

$$S_{(r-1)}(t) = s_r\{\exp -(t/\theta_{r-1})^c\} \int_0^t \frac{[\exp\{-(y/\theta_r)\}^c]^{s_r - 1} c\theta_r^{-1}(y/\theta_r)^{c-1}\exp\{-(y/\theta_r)^c\}}{\exp\{-(y/\theta_{r-1})^c\}} dy$$

$$= s_r \exp\{-(t/\theta_{r-1})^c\} \int_0^t c\theta_r^{-1}(y/\theta_r)^{c-1}\exp[-\{s_r\theta_r^{-c} - \theta_{r-1}^{-c}\}y^c]dy$$

$$= \frac{s_r}{s_r-(\theta_r\theta_{r-1}^{-1})^c}[\exp\{-(t/\theta_{r-1})^c\} - \exp\{-s_r(t/\theta_r)^c\}] \ . \qquad (23.1)$$

If $s_r = (\theta_r/\theta_{r-1})^c$,

$$S_{(r-1)}(t) = s_r\theta_r^{-c}t^c\exp\{-(t/\theta_{r-1})^c\} \ . \qquad (23.2)$$

Again using (21) and (22), we find (assuming $s_r \neq (\theta_r/\theta_{r-1})^c$)

$$S_{(r-2)}(t)$$

$$= s_{r-1}\exp\{-(t/\theta_{r-2})^c\} \circ \{\frac{s_r}{s_r-(\theta_r\theta_{r-1}^{-1})^c}\}^{s_{r-1}} \times$$

$$\times \int_0^t [\exp\{-(y/\theta_{r-1})^c\} - \exp\{-s_r(y/\theta_r)^c\}]^{s_{r-1}-1} \circ$$

$$\circ \frac{cy^{c-1}[\theta_{r-1}^{-c}\exp\{-(y/\theta_{r-1})^c\} - s_r\theta_r^{-c}\exp\{-s_r(y/\theta_r)^c\}]}{\exp\{-(y/\theta_{r-2})^c\}}$$

$$= s_{r-1}\{\frac{s_r}{s_r-(\theta_r\theta_{r-1}^{-1})^c}\}^{s_{r-1}}\exp\{-(t/\theta_{r-2})^c\} \times$$

$$\times \int_0^t cy^{c-1}\sum_{j=0}^{s_{r-1}-1}(-1)^j\binom{s_{r-1}-1}{j}[\theta_{r-1}^{-c}\exp\{-((s_{r-1}-j)\theta_{r-1}^{-c}-js_r\theta_r^{-c}-\theta_{r-2}^{-c})y^c\}$$

$$- s_r\theta_r^{-c}\exp\{-((s_{r-1}-j-1)\theta_{r-1}^{-c} - (j+1)s_r\theta_r^{-c} - \theta_{r-2}^{-c})y^c\}]dy$$

$$= s_{r-1}\{\frac{s_r}{s_r-(\theta_r\theta_{r-1}^{-1})^c}\}^{s_{r-1}}\sum_{j=0}^{s_{r-1}-1}(-1)^j\binom{s_{r-1}-1}{j}[\theta_{r-1}^{-c}\frac{\exp\{-(t/\theta_{r-2})^c\}-\exp(-B_jt^c)}{B_j-\theta_{r-2}^{-c}}$$

$$- s_r\theta_r^{-c}\frac{\exp\{-(t/\theta_{r-2})^c\}-\exp(-B_{j+1}t^c)}{B_{j+1}-\theta_{r-2}^{-c}}] \qquad (24)$$

where $B_j = (s_{r-1}-j)\theta_{r-1}^{-c} - js_r\theta_r^{-c}$ provided $B_j \neq \theta_{r-2}^{-c}$ for any j.

If $B_j = 0$,

$$[\exp\{-(t/\theta_{r-2})^c\} - \exp(-B_j t^c)]/(B_j - \theta_{r-2}^{-c})$$

is replaced by

$$t^c \exp\{-(t/\theta_{r-2})^c\}.$$

The *"non-symmetric"* case requires somewhat more involved computational and technical analysis.

The relevant formula for the survival distribution of the component servicing $S_{h-1}(i_1,\ldots,i_{h-1}) = S_{h-1}(\underset{\sim}{i}(h-1))$ is

$$S_{(h-1),\underset{\sim}{i}(h-1)}(t) = S_{h-1}(t)\int_0^t \{-\frac{d}{dy}\prod_{\substack{\underset{\sim}{u}(h)\\ \in\underset{\sim}{i}(h-1)}} S_{(h),\underset{\sim}{u}(h)}(t)\}\{S_{h-1}(y)\}^{-1}dy \qquad (25)$$

where $\underset{\sim}{u}(h)\in\underset{\sim}{i}(h-1)$ means that $\underset{\sim}{u}(h) \equiv (i_1,i_2,\ldots,i_{h-1},u_h)$.

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Two general forms of multiple stage hierachal replacement are described: one proceeds from replacements over larger subsets to smaller subsets (ultimately individuals), and other is an inverse system. Replacements are from simultaneously aging populations. Some detailed calculations are given for Weibull survival time distributions.

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